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Range Resolution of Targets

GERARD V. TRUNK

Radar Analysis Branch
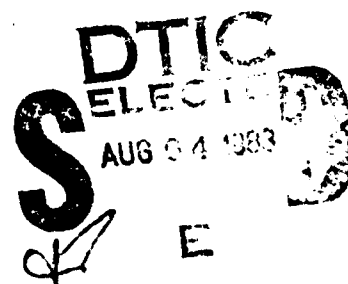*Radar Division*

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Resolving targets in range is formulated as a hypothesis testing problem. A generalized likelihood approach is developed and positive results are obtained. Using a sampling rate of 1.5 samples per pulsewidth, two 20-dB nonfluctuating targets can be resolved at a resolution probability of 0.9 and at a false alarm rate of 0.01, at separations varying between 1/4 and 3/4 of a pulsewidth, depending on the relative phase difference between the targets. The generalized likelihood approach was compared to several easily implemented adhoc approaches, and an ad hoc

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20. ABSTRACT (Continued)

approach involving fitting a pulse shape to the data is only slightly less accurate than the likelihood approach.

CONTENTS

RANGE RESOLUTION OF TARGETS

INTRODUCTION

When attempting to resolve targets, one can either resolve them in angle or range or a combination of both coordinates. The problem of angular resolution has received considerable attention, but the range coordinate is a much more powerful discriminate for separating targets, for example, two targets separated by 1000 ft are separated by two 1-μs pulsewidths but by only $1/20$ of a $2°$ beamwidth at a range of 100 nmi.

The probability of resolving targets is not only a function of target separation but also a function of their signal strengths and phase differences. However, it is usually assumed that if targets are detected, they can be resolved if they are separated by a pulsewidth [1]. In practice this resolution capability is rarely achieved. For instance, a commonly used resolution algorithm which makes no assumptions about the shape of the returned pulse does not attain a resolution probability of 0.9 until the targets are separated by a distance greater than 2.5 pulsewidths [2]. Skolnik [3] conjectures by drawing an analogy with angular resolution that targets within 0.8 pulsewidth of one another should be resolved. The purpose of this report is to generate and evaluate algorithms for achieving resolution when targets are within a pulsewidth of one another.

We will approach the resolution problem as a binary-hypothesis problem where the two hypothesis are:

- H_1: One Target Present

- H_2: Two Targets Present

This hypothesis testing approach to the resolution problem was first considered by Helstrom [4] who investigated the maximum-likelihood detector for the detection of two signals of known separation in time and frequency, but with individual phases and amplitudes and time of arrival of the pair unknown. Later, Allen [5] considered the problem of resolving targets whose positions are known but whose pulse shape may be distorted. Our approach to the problem will be the generalized likelihood ratio test [6] where the unknown parameters are replaced by their maximum likelihood estimates. Root [7] first suggested but later rejected this technique. Although the approach cannot be used for an m-ary hypothesis test (H_i: exactly i targets present) because H_m will always be the most likely hypothesis, it can be used for the binary-hypothesis problem yielding a desired false-alarm rate. Then the m-array hypothesis problem can be attacked by first performing a binary hypothesis as to whether there are one or two targets present. If one decides that two targets are present, one performs another hypothesis test deciding whether two or three targets are present. One continues the testing until the null hypothesis is accepted.

GENERALIZED LIKELIHOOD RATIO TEST—NONCOHERENT APPROACH

We now formulate the resolution problem as a binary-hypothesis test. The received samples are

$$X_i = n_{x_i} + A_1 \frac{\sin (a(R_1 - i))}{a(R_1 - i)} \cos \theta + A_2 \frac{\sin (a(R_2 - i))}{a(R_2 - i)} \cos (\theta + \beta) \text{ and}$$

$$Y_i = n_{y_i} + A_1 \frac{\sin (a(R_1 - i))}{a(R_1 - i)} \sin \theta + A_2 \frac{\sin (a(R_2 - i))}{a(R_2 - i)} \sin (\theta + \beta),$$

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where X_i and Y_i are the inphase and quadrature samples for the ith range cell, n_{x_i} and n_{y_i} are the independent inphase and quadrature Gaussian noise samples with mean zero and variance σ^2, A_1 and A_2 are the target amplitudes, R_1 and R_2 are the target ranges normalized by the sampling interval, θ is the phase of the first target, β is the phase difference between the two targets, $a = 2.7832/\Delta R$, and ΔR is the 3-dB pulsewidth normalized by the sampling interval.

We have assumed that the targets of interest are point targets. This assumption will not affect our results as long as the extent of the target is less than a 1/10 of the pulsewidth. Hence, the results of this analysis do not apply to high-range resolution radars. The range resolution problem is equivalent to deciding which hypothesis is true:

$$H_1: A_1 > 0 \text{ and } A_2 = 0$$

or

$$H_2: A_1 > 0 \text{ and } A_2 = 0.$$

If $\{A_1,A_2,R_1,R_2,\beta\}$ are known, the optimal detector (in the Neyman Pearson sense) given the ith range sample is

$$\Lambda_i = \frac{\frac{1}{2\pi} \int_0^{2\pi} P(X_i,Y_i|H_2) \, d\theta}{\frac{1}{2\pi} \int_0^{2\pi} P(X_i,Y_i|H_1) \, d\theta}$$

where $P(X_i,Y_i|H_j)$ is the joint density of X_i and Y_i given that hypothesis H_j is true. Appendix A shows that the likelihood ratio Λ_i reduces to

$$\Lambda_i = \frac{e^{-Z_i^2/2\sigma^2} I_0\left(\frac{r_i Z_i}{\sigma^2}\right)}{e^{-A_1^2 W_i^2/2\sigma^2} I_0\left(\frac{r_i W_i A_1}{\sigma^2}\right)}, \tag{1}$$

where

$$Z_i^2 = A_1^2 W_i^2 + 2 A_1 A_2 W_i U_i \cos\beta + A_2^2 U_i^2,$$

$$W_i = \begin{cases} \frac{\sin(a(R_1 - i))}{a(R_1 - i)} & |a(R_1 - i)| \leqslant \pi \\ 0 & |a(R_1 - i)| > \pi, \end{cases}$$

$$U_i = \begin{cases} \frac{\sin(a(R_2 - i))}{a(R_2 - i)} & |a(R_2 - i)| \leqslant \pi \\ 0 & |a(R_2 - i)| > \pi, \end{cases}$$

$$r_i = (X_i^2 + Y_i^2)^{1/2},$$

and $I_0(\cdot)$ is the modified Bessel function of the first kind. Hence the likelihood ratio is just the ratio of two Ricean densities; under H_2 there are two signals present and the resultant signal is given by Z_i; under H_1 there is only one signal present with amplitude $W_i A_i$.

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Given the envelope detected samples r_i, $i = 1, \ldots, m$, the optimal test statistic is

$$\Lambda = \prod_{i=1}^{m} \Lambda_i,$$

or equivalently

$$\Lambda' = \sum_{i=1}^{m} \log \Lambda_i.$$

Since in the problem of interest, $\{A_1, A_2, R_1, R_2, \beta, \theta\}$ are unknown, we replace them with their maximum likelihood estimates. Specifically, we find the values of the parameters which minimize

$$L = \sum_{i=1}^{m} (X_i - W_i B - U_i C)^2 + \sum_{i=1}^{m} (Y_i - W_i D - U_i E)^2, \tag{2}$$

where $B = A_1 \cos \theta$, $C = A_2 \cos (\theta + \beta)$, $D = A_1 \sin \theta$, and $E = A_2 \sin (\theta + \beta)$. For any given value of R_1 and R_2, W_i and U_i are known and the maximum likelihood estimates of B, C, D, and E can be found simply by solving the equations

$$\frac{\partial L}{\partial B} = \frac{\partial L}{\partial C} = \frac{\partial L}{\partial D} = \frac{\partial L}{\partial E} = 0.$$

The solutions for the one target case (that is, assuming H_1 is true) are

$$C = E = 0,$$
$$B = \Sigma \ W_i X_i / \Sigma \ W_i^2, \tag{3}$$
$$D = \Sigma \ W_i Y_i / \Sigma \ W_i^2,$$

where all sums run from $i = 1$ to m. The solutions for the two-target case (that is, assuming H_2 is true) are

$$B = [(\Sigma \ U_i^2)(\Sigma \ W_i X_i) - (\Sigma \ W_i U_i)(\Sigma \ U_i X_i)]/\Delta,$$
$$C = [(\Sigma \ W_i^2)(\Sigma \ U_i X_i) - (\Sigma \ W_i U_i)(\Sigma \ W_i X_i)]/\Delta,$$
$$D = [(\Sigma \ U_i^2)(\Sigma \ W_i Y_i) - (\Sigma \ W_i U_i)(\Sigma \ U_i Y_i)]/\Delta,$$
$$E = [(\Sigma \ W_i^2)(\Sigma \ U_i Y_i) - (\Sigma \ W_i U_i)(\Sigma \ W_i Y_i)]/\Delta,$$

where

$$\Delta = (\Sigma \ W_i^2)(\Sigma \ U_i^2) - (\Sigma \ W_i U_i)^2.$$

A direct search is used to estimate the target locations R_1 and R_2. Appendix B outlines this procedure.

The threshold T is set so that

$$Pr\{\Lambda' \geqslant T | H_1\} = P_{FA} = \alpha,$$

where P_{FA} is the probability of false alarm. Since testing for multiple targets is only performed after a target is detected, a fairly high false-alarm rate is permissible. In this report we have selected $\alpha = 0.01$. Because of this high false-alarm rate, Monte Carlo simulations can be used to estimate T. One thousand repetitions were run for the case where the sampling rate was 1.5 samples per pulsewidth, $m = 8$ samples, $\sigma = 1$, $A_1 = 14.14$ (20 dB) and $A_2 = 0$. For each repetition the pulse location was located randomly (uniformly distributed) with respect to the samples. The estimated value for the threshold was 3.7. Essentially, the same threshold was obtained when A_1 was raised to 30 dB. To obtain information on the relationship between resolution and sampling rate, a higher sampling rate of 3.0 samples per pulsewidth will also be considered. For the parameters $m = 10$ samples, $\sigma = 1$, $A_1 = 14.14$ (20 dB), and $A_2 = 0$, the estimated threshold was 3.5

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PROBABILITY OF RESOLUTION

The probability of resolving two 20-dB nonfluctuating targets at a false-alarm rate of $\alpha = 0.01$ was found by simulation and Figs. 1 and 2 show the results as a function of target separation for $\beta = 0°$, 45°, 90°, 135°, and 180°. Two targets are resolvable if they are separated by 1/4 to 3/4 pulsewidths depending on their phase relationship. In general, it is easier to separate targets if they are out of phase (that is, β close to 180°). The reversal of the 135° and 180° curves is caused by the fact that the resolution test between one and two targets is not applied until after a target is detected. When the targets are 180° out of phase, there is a large target cancellation. In fact the probability of resolution curve is practically identical with the probability of detection curve; that is, when two 20-dB targets are 180° out of phase, they can be resolved as soon as they can be detected. The dashed curve in Fig. 2 represents the resolution capability of deciding between one or two targets when the detection criterion is not imposed. As expected this curve shows that it is easier to resolve targets when $\beta = 180°$ than when $\beta = 135°$.
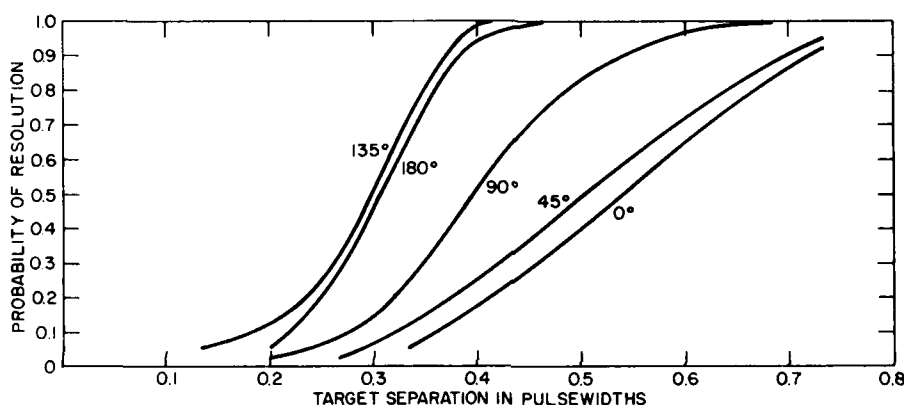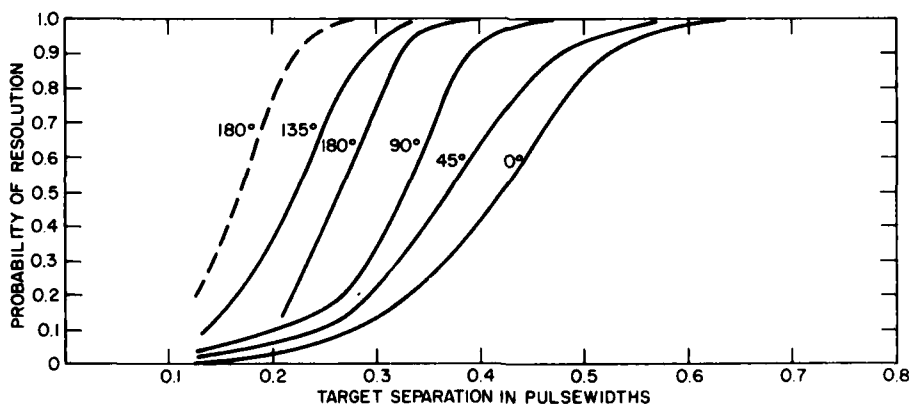

Fig. 1 — Probability of resolution as a function of range separation: Noncoherent likelihood procedure; sampling rate $\Delta R = 1.5$ samples per pulsewidth; target strengths—nonfluctuating, $A_1 = A_2 = 20$ dB; phase differences $= 0°, 45°, 90°, 135°,$ and 180°.

Fig. 2 — Probability of resolution as a function of range separation: Noncoherent likelihood procedure; sampling rate $\Delta R = 3.0$ samples per pulsewidth; target strengths—nonfluctuating, $A_1 = A_2 = 20$ dB; phase differences $= 0°, 45°, 90°, 135°,$ and 180°.

Next, we investigated the effects of unequal target strengths. Figures 3 and 4 show the results for $A_1 = 20$ dB and $A_2 = 30$ dB. Figures 5 and 6 show the results for $A_1 = 20$ dB and $A_2 = 13$ dB. Comparing Figs. 1, 3, and 5, and Figs. 2, 4, and 6, there appears to be less variation with phase difference when the targets have unequal target strengths. The $A_1 = 20$ dB and $A_2 = 30$ dB cases fall in the middle of the $A_1 = A_2 = 20$ dB cases, whereas the $A_1 = 20$ dB and $A_2 = 13$ dB cases fall to the right (worse performance) of the $A_1 = A_2 = 20$ dB cases.

I next investigated the effects of sampling rate. In addition to the 1.5 and 3.0 samples-per-pulsewidth case, I investigated 1.0, 2.0 and 4.0 samples-per-pulsewidth. Repeating the procedure for estimating the threshold, thresholds of 4.3 ($m = 7$), 3.1 ($m = 8$) and 3.5 ($m = 14$) were found for a false-alarm rate of $\alpha = 0.01$ for the 1.0, 2.0, and 4.0 samples-per-pulsewidth cases. The probability of resolution curves are shown in Figs. 7, 8 and 9. Table 1 gives the range separation necessary to achieve a 0.5-resolution probability. As the sampling rate is increased from 1 to 4 samples per pulsewidth (a 400% increase), the corresponding decrease in range separation averages only 41%. Therefore, while increasing the sampling rate improves resolution, it does not appear that it is worth the cost of faster A/D converters and increased memory and processing. The best compromise appears to be either 1.5 or 2.0 samples per pulsewidth.
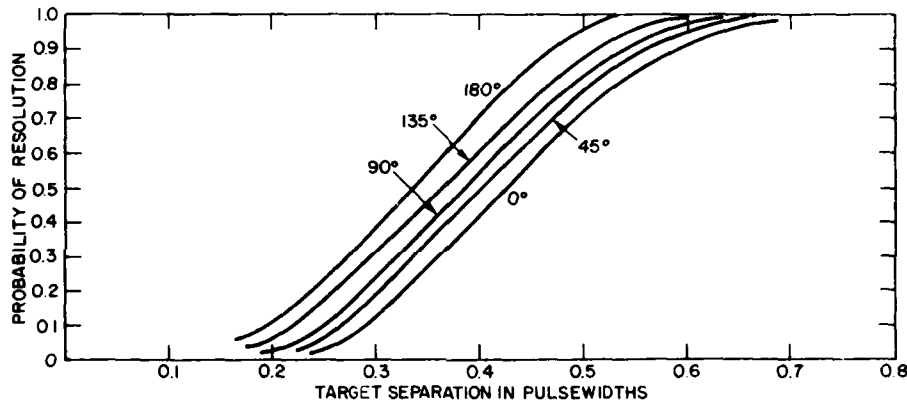

Fig. 3 — Probability of resolution as a function of range separation: Noncoherent likelihood procedure; sampling rate $\Delta R = 1.5$ samples per pulsewidth; target strengths—nonfluctuating, $A_1 = 20$ dB and $A_2 = 30$ dB; phase differences $= 0°$, $45°$, $90°$, $135°$, and $180°$.
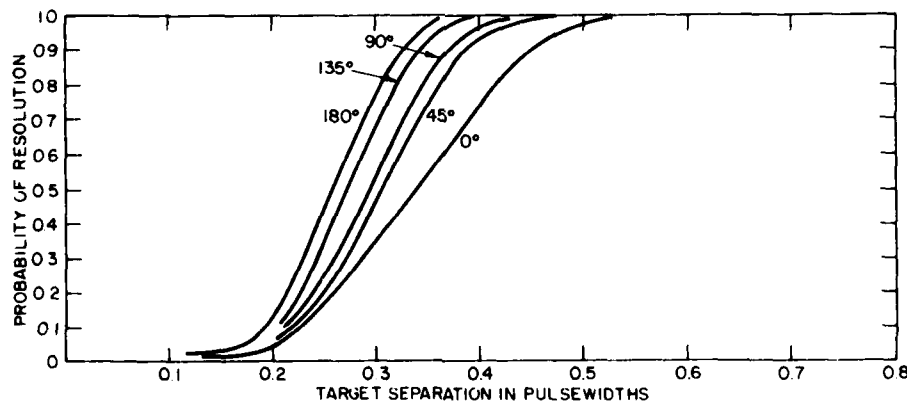

Fig. 4 — Probability of resolution as a function of range separation: Noncoherent likelihood procedure; sampling rate $\Delta R = 3.0$ samples per pulsewidth; target strengths—nonfluctuating, $A_1 = 20$ dB and $A_2 = 30$ dB; phase differences $= 0°$, $45°$, $90°$, $135°$, and $180°$.
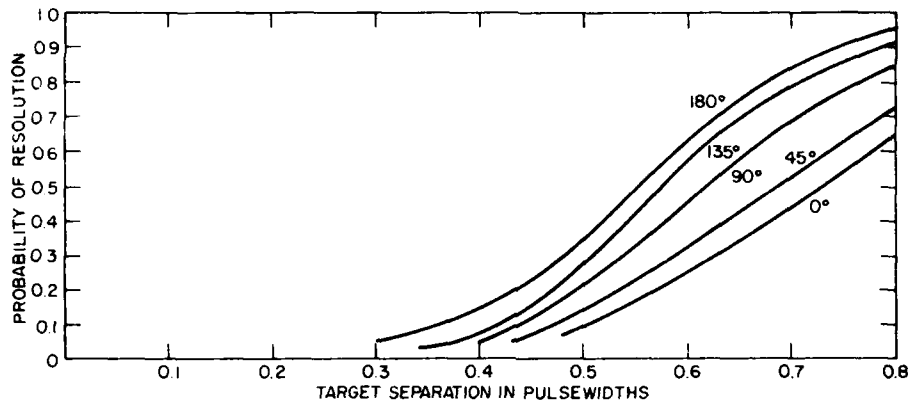
Fig. 5 — Probability of resolution as a function of range separation: Noncoherent likelihood procedure; sampling rate $\Delta R = 1.5$ samples per pulsewidth; target strengths—nonfluctuating, $A_1 = 20$ dB and $A_2 = 13$ dB; phase differences $= 0°$, $45°$, $90°$, $135°$, and $180°$.

Fig. 6 — Probability of resolution as a function of range separation: Noncoherent likelihood procedure; sampling rate $\Delta R = 3.0$ samples per pulsewidth; target strengths—nonfluctuating, $A_1 = 20$ dB and $A_2 = 13$ dB; phase differences $= 0°$, $45°$, $90°$, $135°$, and $180°$.
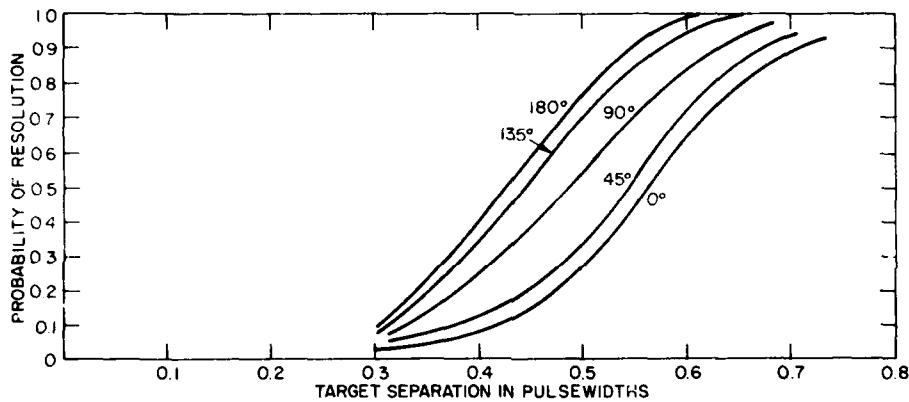

Fig. 7 — Probability of resolution as a function of range separation. Noncoherent likelihood procedure; sampling rate $\Delta R = 1.0$ samples per pulsewidth; target strengths—nonfluctuating, $A_1 = A_2 = 20$ dB; phase differences $= 0°$, $45°$, $90°$, $135°$, and $180°$.
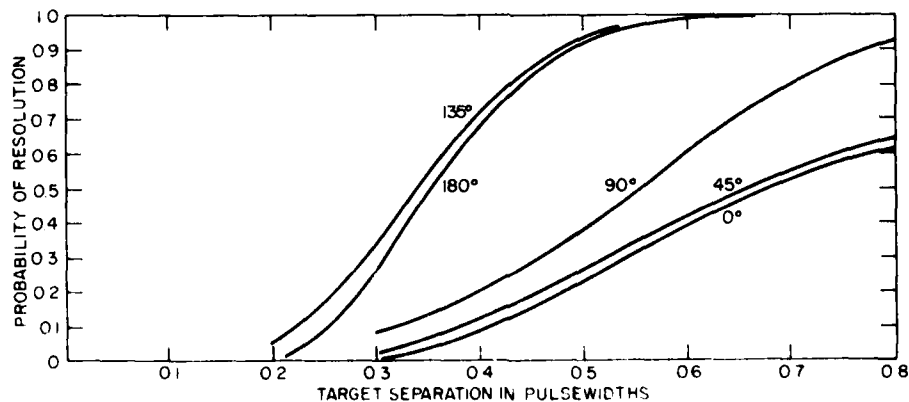
Fig. 8 — Probability of resolution as a function of range separation: Noncoherent likelihood procedure; sampling rate ΔR = 2.0 samples per pulsewidth; target strengths—nonfluctuating. $A_1 = A_2$ = 20 dB; phase differences = 0°, 45°, 90°, 135°, and 180°.
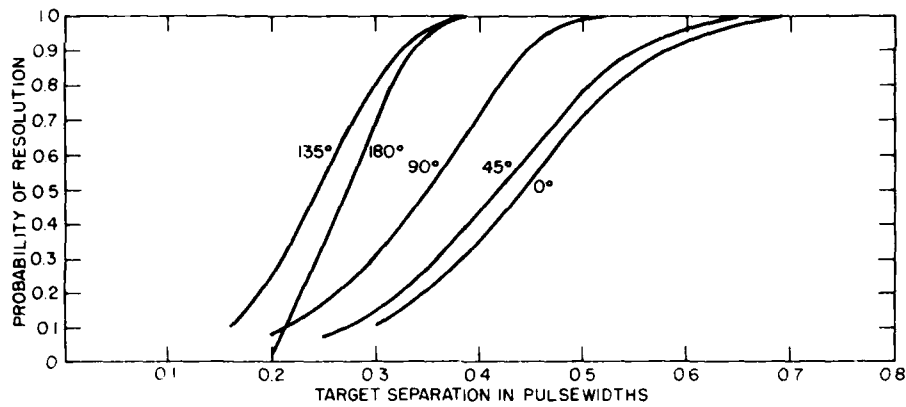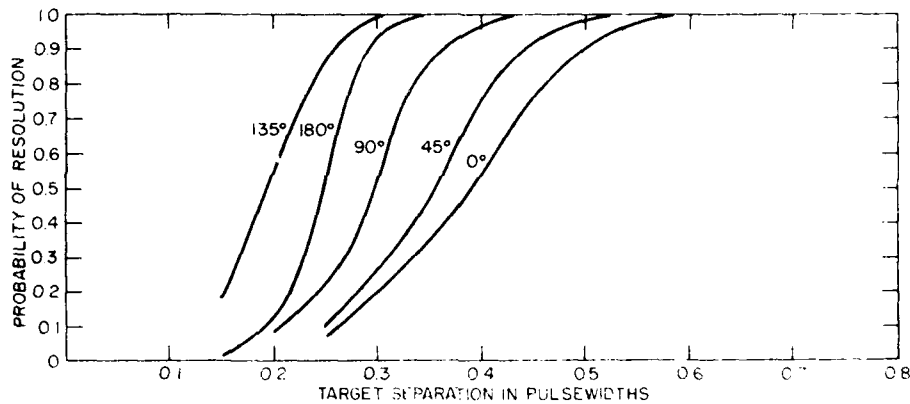

Fig. 9 — Probability of resolution as a function of range separation: Noncoherent likelihood procedure; sampling rate ΔR = 4.0 samples per pulsewidth; target strengths—nonfluctuating. $A_1 = A_2$ = 20 dB; phase differences = 0°, 45°, 90°, 135°, and 180°.

Table 1 — Range Separation Required to achieve
a Probability of Resolution = 0.5

Phase Difference (deg)	Number of Samples per Pulsewidth				
	(1.0)	(1.5)	(2.0)	(3.0)	(4.0)
180	0.36	0.31	0.28	0.27	0.25
135	0.34	0.30	0.25	0.23	0.20
90	0.56	0.40	0.36	0.33	0.30
45	0.66	0.51	0.42	0.37	0.36
0	0.68	0.54	0.45	0.42	0.39

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MULTIPLE PULSES

I now consider the case of multiple pulses on target. There are several situations that one can consider. These situations depend upon whether the parameters $\{A_1, A_2, R_1, R_2, \theta, \beta\}$ remain constant or change between pulses. Attention will be restricted to the situation when pulse-to-pulse frequency agility is used. In this case it is reasonable to assume that R_1 and R_2 remain constant for all pulses but that A_1, A_2, θ, and β are independent per pulse. If X_{ij} and Y_{ij} are the jth time samples of the ith range sample, it is straightforward to show that B_j and C_j can be found from Eqs. (3) and (4) by replacing X_i with X_{ij} and similarly D_j and E_j can be found from Eqs. (3) and (4) by replacing Y_i with Y_{ij}.

The total likelihood ratio is just the product of the individual likelihood ratios. The thresholds for $\alpha = 0.01$ and a 20-dB nonfluctuating target for $n = 1, 2, 4,$ and 8 pulses were found by simulation and are given in Table 2.

Table 2 — Threshold Yielding a Probability
of False Resolution = 0.01

Number of Pulses	Sampling Rate	
	1.5 ($m = 8$)	3.0 ($m = 10$)
$n = 1$	3.7	3.5
$n = 2$	4.7	5.0
$n = 4$	6.0	6.3
$n = 8$	8.5	10.0

Figures 10 and 11 show the results for two 20-dB Rayleigh fluctuating targets. The phase difference was uniformly distributed between 0 and 180°. The reason for the poor performance of the one-pulse case is because the probability of detection is only 0.9 for a 20-dB Rayleigh fluctuating target. Tables 3 and 4 give the ranges at which two targets can be resolved at 0.5 and 0.9 probabilities when multiple pulses are available.

Fig. 10 — Probability of resolution as a function of range separation: Noncoherent likelihood procedure; sampling rate $\Delta R = 1.5$ samples per pulsewidth; target strengths(emfluctuating, $A_1 = A_2$ = 20 dB; phase difference are independent and uniformly distributed per pulse; number of pulses $N = 1, 2, 4,$ and 8.
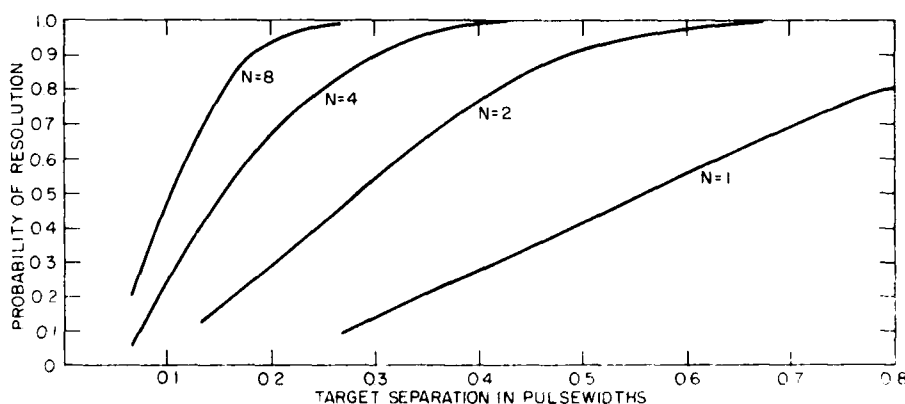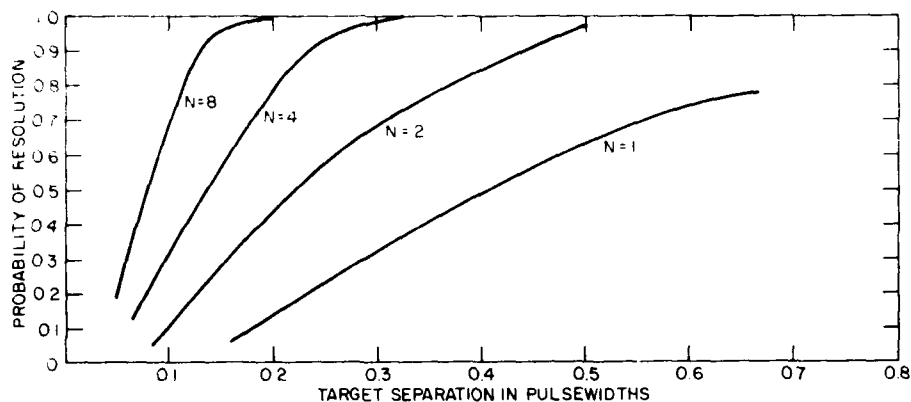
Fig. 11 — Probability of resolution as a function of range separation: Noncoherent likelihood procedure; sampling rate ΔR = 3.0 samples per pulsewidth; target strengths (emfluctuating, $A_1 = A_2$ = 20 dB; phase difference are independent and uniformly distributed per pulse, number of pulses N = 1, 2, 4, and 8.

Table 3 — Range Separation Required to Achieve
a Probability of Resolution = 0.5

Number of Pulses	Number of Samples per Pulsewidth	
	(1.5)	(3.0)
1	0.57	0.42
2	0.28	0.22
4	0.15	0.14
8	0.10	0.09

Table 4 — Range Separation Required to Achieve
a Probability of Resolution = 0.9

Number of Pulses	Number of Samples per Pulsewidth	
	(1.5)	(3.0)
2	0.50	0.45
4	0.30	0.23
8	0.18	0.13

AD HOC NONCOHERENT PROCEDURES

The generalized likelihood test involves a 6-parameter minimization implemented by a two dimensional search (see Appendix B) and consequently, is very time consuming. Therefore, I investigated a combination of 2 adhoc approaches. The first approach involves the generation of double peaks and the second involves two small of a maximum amplitude for a fixed number of samples above the detection threshold.

Figure 12 shows an example of a *double peak*. To have a double peak it is necessary for the amplitudes to increase (first peak), decrease (null), and increase (second peak) again. Furthermore,
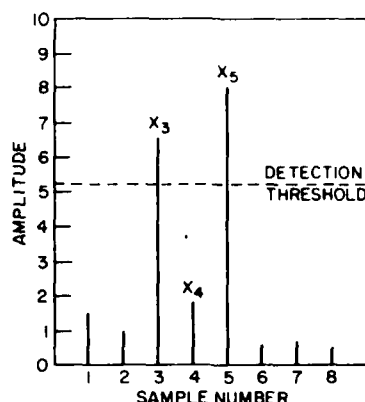
Fig. 12 — Example of a double peak

both peaks must be above the detection threshold. There is no condition on the null between the two peaks. When the sampling rate is 1.5 samples per pulsewidth, a double peak is a good indication of two targets. For instance, in 10,000 repetitions of a 20-dB target, no double peaks were observed. On the other hand, if the sampling rate is 3.0 samples per pulsewidth, a double peak is no longer a good indication of two targets. For instance, in 10,000 repetitions of a 20-dB target, 442 double peaks were observed. Since this corresponds to a false-alarm rate of 0.044, the procedure was modified. A *modified double peak* is the same as a double peak except that there is an additional condition on the null depth. Specifically, the null depth ΔN defined by

$$\Delta N = \text{Minimum \{First Peak} - \text{Null Value, Second Peak} - \text{Null Value\}}$$

must be greater than a specified value. The required null depth thresholds were found by simulation and Table 5 gives the results. When there are multiple pulses, two targets are declared if any of the pulses contains a modified double peak.

A double peak is most likely to occur when the two targets are 180° out of phase. On the other hand, if the targets are in phase, one would expect a small peak value for a given number of samples greater than the detection threshold. Simulations were run to generate histograms of peak return for a given number of samples above the detection threshold. For instance, for 10,000 repetitions of a 20-dB Rayleigh fluctuating target, 1302 times there were no samples above the detection threshold, 1210 times there was one sample above the threshold, 4703 times there were two samples above the threshold, 2783 times there were three samples above the threshold, and one time four samples were above the threshold. Figure 13 shows the amplitude histogram when there are two samples above the threshold. If the resolution threshold is set at 5.95 a false-alarm rate approximately equal to 0.01 is obtained. To obtain the resolution threshold when there are four samples above the detection threshold, the simulation was repeated with a 30-dB fluctuating target. When there are multiple pulses on target, the pulses are integrated noncoherently; then the same procedure is applied to the noncoherent sums. The results for 1.5 samples and 3.0 samples per pulsewidth are given in Tables 6 and 7 respectively.

Using the resolution thresholds of Tables 6 and 7, the probability of resolution curves were generated and are shown in Figs. 14 and 15. Comparing the adhoc results with the likelihood approach results appearing in Figs. 10 and 11, one observes that the likelihood approach is superior; in fact, the likelihood approach for 2 pulses at a sampling rate of 1.5 samples per pulsewidth is approximately the same as the adhoc approach for 8 pulses at a sampling rate of 3.0 samples per pulsewidth.

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Table 5 — Thresholds for Null Depth
Necessary to Achieve a False Alarm Probability
of $\alpha = 0.01$

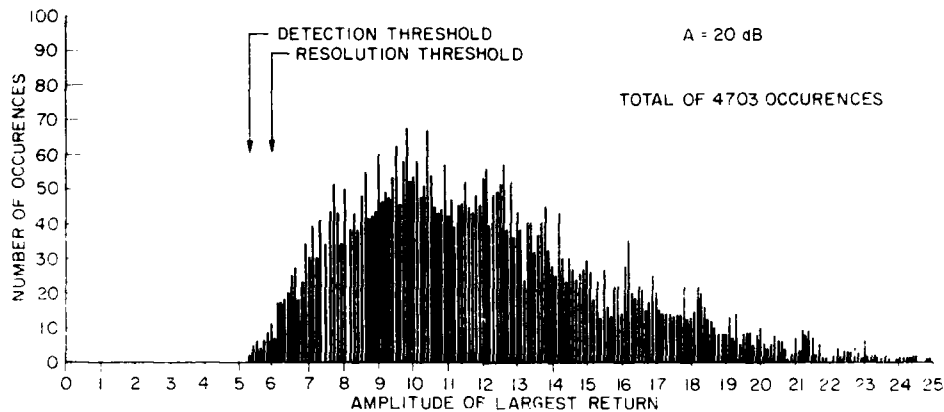| Number of Pulses	Null Depth Threshold Values	
	Number of Samples per Pulsewidth	
	(1.5)	(3.0)
1	0	1.25
2	0	1.40
4	0	1.60
8	0	2.00

Fig. 13 — Amplitude histogram of case where there are two pulses about detection threshold: $A_1 = 20$ dB and $A_2 = -\infty$ dB.

Table 6 — Resolution Threshold Yielding
a False-Alarm Rate of 0.01 when the Sampling Rate
is 1.5 Samples per Pulsewidth

Number of Pulses	Number of Samples Above Threshold			
	(2)	(3)	(4)	(5 and above)
1	5.95	7.35	33.65	∞
2	9.82	15.32	51.42	∞
4	19.48	28.78	87.88	∞
8	50.55	62.55	100.05	∞

Table 7 — Resolution Threshold Yielding a False-Alarm Rate
of 0.01 when the Sampling Rate is 3.0 Samples per Pulsewidth

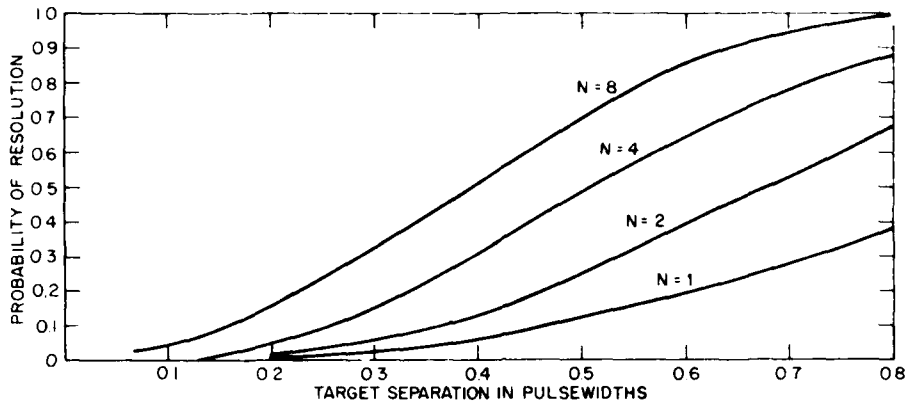| Number of Pulses	Number of Samples Above Threshold						
	(2)	(3)	(4)	(5)	(6)	(7)	(8 and above)
1		5.65	6.45	8.85	15.05	46.75	∞
2		8.92	10.82	14.92	24.32	56.42	∞
4			19.28	26.88	36.38	100.88	∞
8				55.25	68.05	200.05	∞

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Fig. 14 — Probability of resolution as a function of range separation: Noncoherent likelihood pro-
cedure; sampling rate ΔR = 1.5 samples per pulsewidth; target strengths(emfluctuating, $A_1 = A_2$
= 20 dB; phase difference are independent and uniformly distributed per pulse, number of pulses
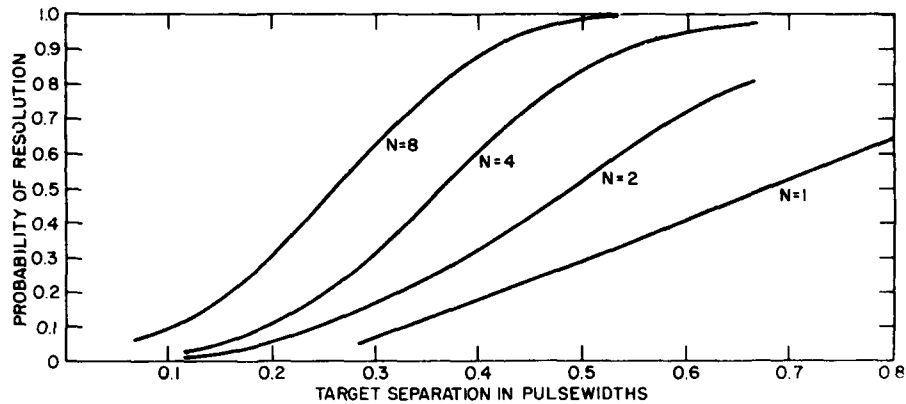N = 1, 2, 4, and 8.

Fig. 15 — Probability of resolution as a function of range separation: Noncoherent likelihood pro-
cedure; sampling rate ΔR = 3.0 samples per pulsewidth; target strengths(emfluctuating, $A_1 = A_2$
= 20 dB; phase difference are independent and uniformly distributed per pulse, number of pulses
N = 1, 2, 4, and 8.

Table 8 — Threshold for
Coherent Likelihood Ratio with
a Sampling Rate of
1.5 Samples per Pulsewidth

Number of Pulses	Threshold
1	12.4
2	18.5
4	26.7
8	39.0

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GENERALIZED LIKELIHOOD RATIO TEST—COHERENT APPROACH

If all the signal parameters $(A_1, A_2, R_1, R_2, \beta)$ except θ are known, the optimal test is given by the likelihood ratio (Eq. (1)) involving the noncoherent samples. Since the signal parameters are unknown, we estimated them and substituted the estimates into the noncoherent likelihood ratio. However, since the likelihood ratio of noncoherent samples now contains estimates instead of the true parameters, it is no longer optimal, and it is possible that the likelihood ratio based on coherent samples will yield better results. Consequently, we now consider the coherent likelihood ratio.

The coherent likelihood ratio for n pulses is

$$\Lambda = \prod_{j=1}^{n} \frac{\prod_{i=1}^{m}\left[\frac{1}{2\pi} \exp\left\{-\frac{1}{2}(X_{ij} - W_i B_j - U_i C_j)^2 + (Y_{ij} - W_i D_j - U_i E_j)^2]\right\}\right]}{\prod_{i=1}^{m}\left[\frac{1}{2\pi} \exp\left\{-\frac{1}{2}[(X_{ij} - W_i B_j)^2 + (Y_{ij} - W_i D_j)^2]\right\}\right]}$$

Equation (4) gives the maximum likelihood estimates for the two target signal parameters (B_j, C_j, D_j, E_j) appearing in the numerator, and Eq. (3) gives the maximum likelihood estimates for the single target signal parameters (B_j, D_j) appearing in the denominator. Taking the log of the coherent likelihood yields as a test statistic, the square error residue (Eq. (2)) evaluated for one target minus the square error residue (Eq. (2)) evaluated for two targets

$$\log \Lambda = \underset{R_1, B_j, D_j}{\text{Min}} \{L\} - \underset{R_1, R_2, B_j, C_j, D_j, E_j}{\text{Min}} \{L\}. \tag{5}$$

We now proceed as we did for the noncoherent likelihood. We first calculate the thresholds for multiple pulses by using simulation techniques. The thresholds for $n = 1, 2, 4,$ or 8 pulses are given in Table 8. Next, again using simulation, Figs. 16-19 show the resolution curves for $A_1 = A_2 = 20$ dB, $A_1 = 20$ dB and $A_2 = 30$ dB, $A_1 = 20$ dB and $A_2 = 13$ dB, and multiple pulses. Comparing the coherent results with the noncoherent results of Figs. 1, 3, 5, and 10, one concludes that the coherent results are uniformly better and are much better when the targets have different signal strengths.
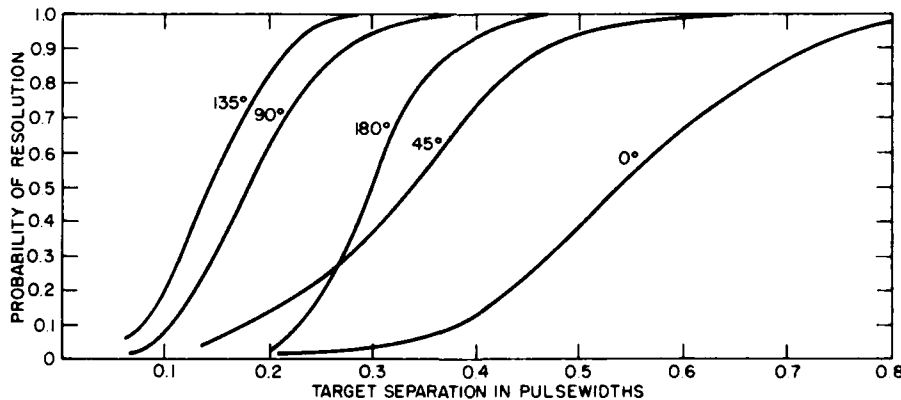

Fig. 16 — Probability of resolution as a function of range separation: Coherent likelihood procedure; sampling rate $\Delta R = 1.5$ samples per pulsewidth; target strengths—nonfluctuating, $A_1 = A_2 = 20$ dB; phase differences $= 0°, 45°, 90°, 135°,$ and $180°$.
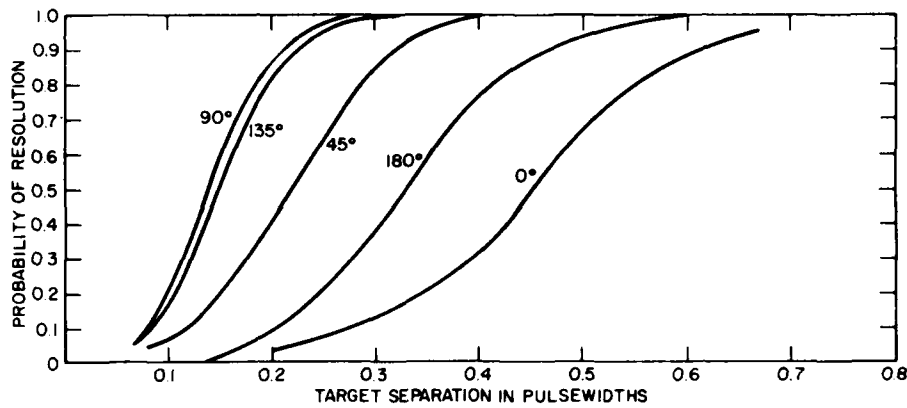
Fig. 17 — Probability of resolution as a function of range separation: Coherent likelihood procedure; sampling rate ΔR = 1.5 samples per pulsewidth; target strengths—nonfluctuating, A_1 = 20 dB and A_2 = 30 dB; phase differences = 0°, 45°, 90°, 135°, and 180°.
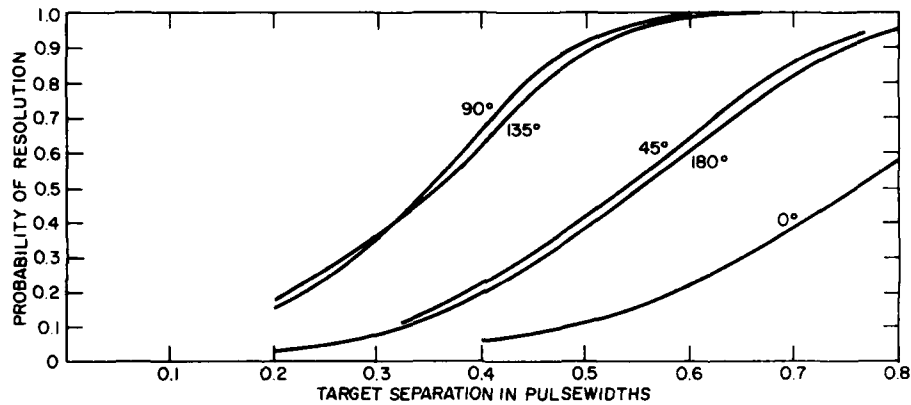

Fig. 18 — Probability of resolution as a function of range separation: Coherent likelihood procedure; sampling rate ΔR = 1.5 samples per pulsewidth; target strengths—nonfluctuating, A_1 = 20 dB and A_2 = 13 dB; phase differences = 0°, 45°, 90°, 135°, and 180°.
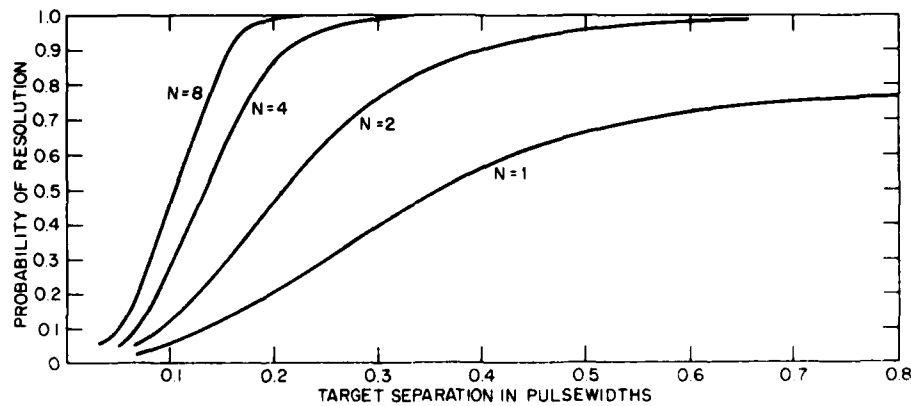

Fig. 19 — Probability of resolution as a function of range separation: Coherent likelihood procedure; sampling rate ΔR = 1.5 samples per pulsewidth; target strengths(emfluctuating, A_1 = A_2 = 20 dB; phase difference are independent and uniformly distributed per pulse, number of pulses N = 1, 2, 4, and 8.

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AD HOC COHERENT PROCEDURE

Although the previously discussed ad hoc noncoherent procedures are not very applicable to the coherent data, the coherent likelihood ratio suggests another ad hoc approach: instead of using the difference in square errors given by Eq. (5), use the square error associated with fitting one target to the data. If only one target is present, the residue should only be noise and hence should be small. If two or more targets are present, the residue will contain signal and should be large. This statistic only requires a one-dimensional search and can be accomplished very quickly. In fact since the initialization procedure given in Appendix B is fairly accurate, it is unclear whether or not the one-dimensional search is required.

Using simulation techniques, the appropriate thresholds were calculated and are given in Table 9. Next, again using simulation, the resolution curves for $A_1 = A_2 = 20$ dB, $A_1 = 20$ dB and $A_2 = 30$ dB, $A_1 = 20$ dB and $A_2 = 13$ dB, and multiple pulses were calculated and are shown in Figs. 20 to 23. Comparing the ad hoc coherent results with the coherent likelihood results shown in Figs. 16 to 19, one concludes that the coherent likelihood results are only slightly better. Consequently, because of the reduced calculations required, I would recommend the ad hoc approach.

Table 9 — Thresholds for
Fitting Single Target to the Data
with a Sampling Rate of
1.5 Samples per Pulsewidth

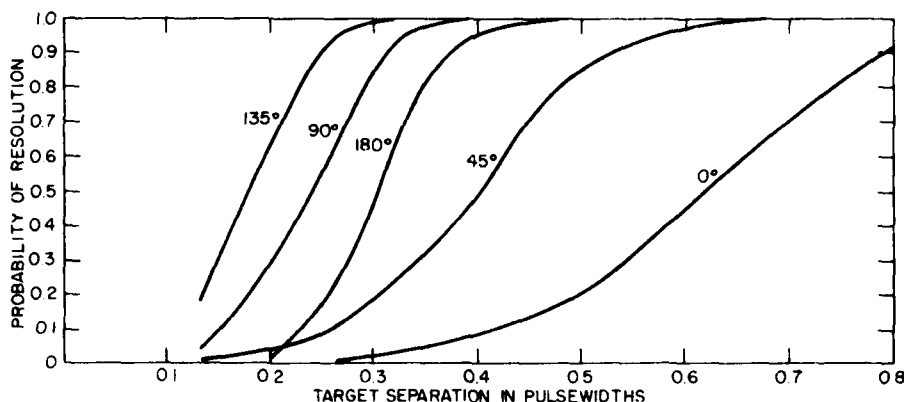| Number of Pulses	Threshold
1	28.5
2	44.7
4	73.7
8	148.4

Fig. 20 — Probability of resolution as a function of range separation: Coherent likelihood procedure; sampling rate $\Delta R = 1.5$ samples per pulsewidth; target strengths—nonfluctuating, $A_1 = A_2 = 20$ dB; phase differences = $0°$, $45°$, $90°$, $135°$, and $180°$.

Fig. 21 — Probability of resolution as a function of range separation: Coherent likelihood procedure; sampling rate $\Delta R = 1.5$ samples per pulsewidth; target strengths—nonfluctuating, $A_1 = 20$ dB and $A_2 = 30$ dB; phase differences $= 0°, 45°, 90°, 135°,$ and $180°$.
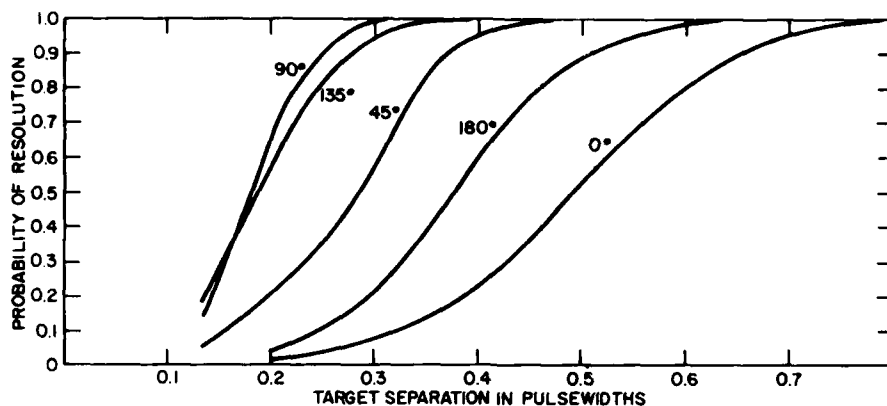

Fig. 22 — Probability of resolution as a function of range separation: Coherent likelihood procedure; sampling rate $\Delta R = 1.5$ samples per pulsewidth; target strengths—nonfluctuating, $A_1 = 20$ dB and $A_2 = 13$ dB; phase differences $= 0°, 45°, 90°, 135°,$ and $180°$.
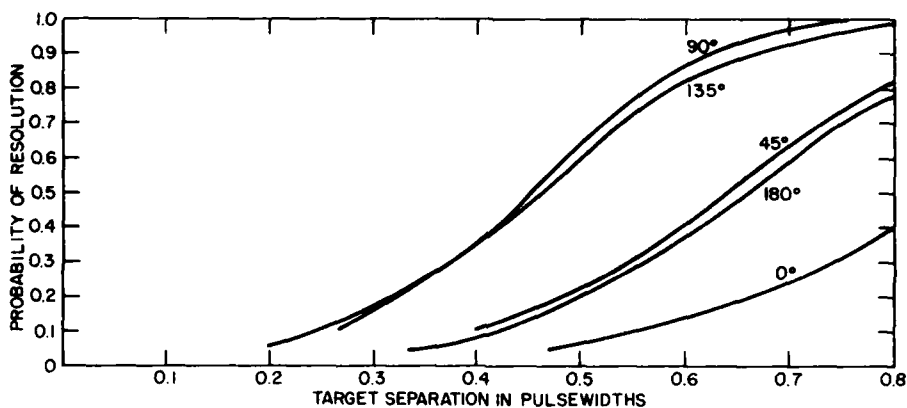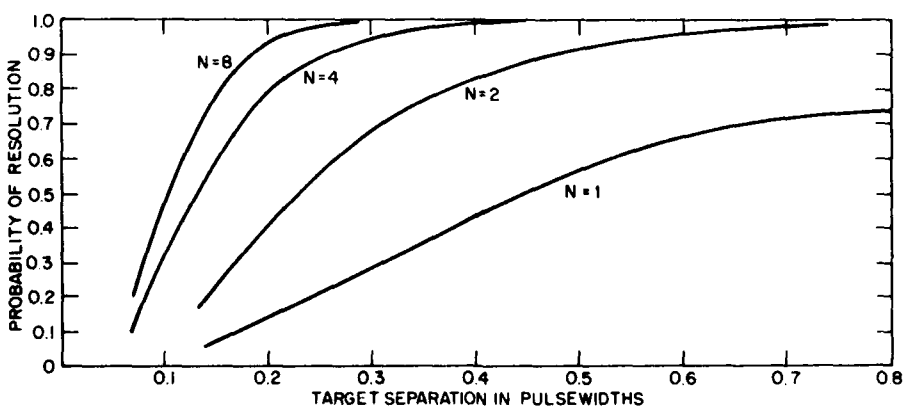

Fig. 23 — Probability of resolution as a function of range separation: Coherent likelihood procedure; sampling rate $\Delta R = 1.5$ samples per pulsewidth; target strengths(emfluctuating, $A_1 = A_2 = 20$ dB; phase difference are independent and uniformly distributed per pulse, number of pulses $N = 1, 2, 4,$ and 8.

SUMMARY

The problem of resolving targets in range has been formulated as a hypothesis-testing problem where the unknown parameters (target's location, amplitude, and phase) are replaced by their least-square estimates. When the likelihood ratio test using either coherent or noncoherent samples is applied to the range samples surrounding a detection, good resolution results are obtained. The probability of resolution is a function of the sampling rate, target amplitudes, target separation, and target phase difference. Using a sampling rate of 1.5 samples per pulsewidth, two 20-dB nonfluctuating targets can be resolved at a resolution probability of 0.9 at separations varying between 1/4 and 3/4 of pulsewidth depending on the relative phase difference between the two targets. This result can be improved further by processing multiple pulses. The coherent likelihood is preferred because it yields better results when the two targets have unequal strengths.

The likelihood approach was compared to an easily implemented ad hoc approach. The ad hoc approach declared the presence of two targets if either of two conditions were met: (1) the return signal was double peaked with both peaks above the detection threshold or (2) the peak amplitude was too small in relationship to the number of samples above the detection threshold.

Unfortunately, while this ad hoc approach is easily implemented, the likelihood approach has superior performance. A slightly more complicated approach involves fitting a pulse shape to the data and comparing the residue error to an appropriate threshold. This procedure is only slightly less accurate than the coherent likelihood ratio and hence is the recommended procedure.

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Appendix A
CALCULATION OF THE LIKELIHOOD RATIO Λ

The likelihood ratio is

$$\Lambda = \frac{<p(x,y|H_2)>_\theta}{<p(x,y|H_2)>_\theta}$$

where the subscript for the ith sample has been dropped and the notation $<\cdot>_\theta$ indicates that the density is averaged over θ. We now calculate $<p(x,y|H_2)>_\theta$ and obtain $<p(x,y|H_1)>_\theta$ from it by setting the amplitude A_2 of the second signal to zero.

The inphase and quadrature signals x and y can be written as

$$x = n_x + A_1 W \cos\theta + A_2 U \cos(\theta + \beta)$$
$$y = n_y + A_1 W \sin\theta + A_2 U \sin(\theta + \beta)$$

where

$$W = \begin{cases} \dfrac{\sin(a(R_1 - i))}{a(R_1 - i)} & |a(R_1 - i)| \leqslant \pi \\[2em] 0 & |a(R_1 - i)| > \pi \end{cases}$$

$$U = \begin{cases} \dfrac{\sin(a(R_2 - i))}{a(R_2 - i)} & |a(R_2 - i)| \leqslant \pi \\[2em] 0 & |a(R_2 - i)| > \pi \end{cases}$$

Since x and y are independent, $<p(x,y|H_2)>_\theta$ can be written as

$$<p(x,y|H_2)>_\theta = \frac{1}{2\pi\sigma^2} \exp\{ - [x - A_1 W \cos\theta - A_2 U \cos(\theta + \beta)]^2/2\sigma^2$$

$$- [y - A_1 W \sin\theta - A_2 U \sin(\theta + \beta)]^2/2\sigma^2\}>_\theta.$$

Performing the squaring operation and combining terms yields

$$<p(x,y|H_2)>_\theta = \frac{1}{2\pi\sigma^2} \exp\{ - [x^2 + y^2 + A_1^2 W^2 + 2A_1 A_2 WU \cos\beta$$

$$+ A_2^2 U^2]/2\sigma^2\} < \exp\{\frac{1}{\sigma^2} [A_1 Wx \cos\theta$$

$$+ A_1 Wy \sin\theta + A_2 Ux \cos(\theta + \beta) + A_2 Uy \sin(\theta + \beta)]\}>_\theta$$

Letting $x = r \cos\phi$ and $y = r \sin\phi$ yields

$$<p(r,\phi|H_2)>_\theta = C \quad \exp\{\frac{r}{\sigma^2} [A_1 W \cos(\theta - \phi) + A_2 U \cos(\theta + \beta - \phi)]\}>_\theta$$

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where

$$C = \frac{r}{2\pi\sigma^2} \exp\{-[r^2 + A_1^2 W^2 + 2A_1 A_2 WU \cos\beta + A_2^2 U^2]/2\sigma^2\}$$

Since the integral over θ is from 0 to 2π, the value of the integral is independent of ϕ and thus we can set $\phi = 0$:

$$<p(r,\phi|H_2)>_\theta = C<\exp\{r[A_1 W \cos\theta + A_2 U \cos(\theta + \beta)]/\sigma^2\}>_\theta$$

Expanding $\cos(\theta + \beta)$ and rearranging yields

$$<p(r,\phi|H_2)>_\theta = C<\exp\{r[A_1 W + A_2 U \cos\beta)\cos\theta - A_2 U \sin\beta \sin\theta]/\sigma^2\}>_\theta$$

Letting $(A_1 W + A_2 U \cos\beta) = Z \cos\alpha$ and $-A_2 U \sin\beta = Z \sin\alpha$ yields

$$<p(r,\phi|H_2)>_\theta = C<\exp\{rZ \cos(\theta - \alpha)/\sigma^2\}>_\theta$$

The preceding expression is independent of α. Setting $\alpha = 0$ yields an integral which is a modified Bessel function:

$$<p(r,\phi|H_2)>_\theta = \frac{r}{2\pi\sigma^2} \exp\{-[r^2 + Z^2]/2\sigma^2\} I_0(rz/\sigma^2)$$

where $Z^2 = A_1^2 W^2 + 2A_1 A_2 WU \cos\beta + A_2^2 U^2$.

Integrating over ϕ and substituting into the likelihood ratio yields

$$\Lambda = \frac{\exp\{-Z^2/2\sigma^2\} I_0(rZ/\sigma^2)}{\exp\{-A_1^2 W^2/2\sigma^2\} I_0(rA_1 W/\sigma^2)}.$$

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Appendix B
SEARCH PROCEDURE

The square error is minimized by using a direct search[*] for the target locations. The initial starting location for the first target is found by interpolating around the largest return. To do the necessary interpolation, we assume that the pulse shape can be approximated by a Gaussian-shaped pattern

$$A(R) = e^{-\alpha(R-R_T)^2},$$

where $A(R)$ is the pulse gain at range R when a point target is located at range R_T. The appropriate normalizing constant α can be found from

$$0.5 = [e^{-\alpha(R_{3dB}/2)^2}]^2,$$

yielding

$$\alpha = -\frac{2\ln(.5)}{R_{3dB}^2} = \frac{1.386}{R_{3dB}^2}.$$

Now, assume that the target is detected and the corresponding ranges to the largest return and next largest adjacent return are R_1 and $R_1 + \Delta R$ where ΔR is the sampling rate. Then, the corresponding amplitude returns are given by

$$A_1 = e^{-\alpha(R_1-R_T)^2}$$

and

$$A_2 = e^{-\alpha(R_1+\Delta R-R_T)^2}.$$

Solving these equations for the target range yields

$$R_T = R_1 + \frac{\Delta R}{2} + \frac{1}{2\alpha(\Delta R)}\log(A_2/A_1).$$

However, the above estimate neeo not lie between R_1 and $R_1 + \Delta R$. Consequently, if

$$\hat{R}_T < R_1 \text{ set } \hat{R}_T = R_1;$$

and if

$$\hat{R}_T > R_1 + \Delta R, \text{ set } \hat{R}_T = R_1 + \Delta R.$$

To estimate the starting location for two targets we first use the previous procedure to find the location of the first target. We then noncoherently subtract out the first target and apply the previous procedure to the residue.

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[*]R. *Hooke* and T. Jeeves, "Direct search solution of numerical and statistical problems," *J. ACM*, **8**, 212 (1961).